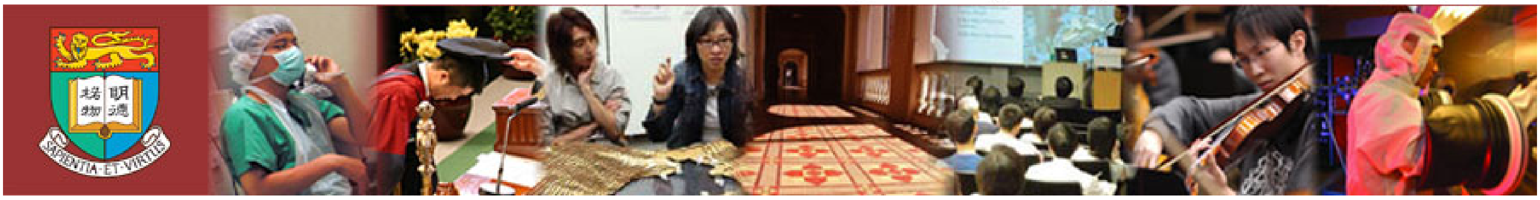

Stability and Dissipativity Analysis of Distributed Delay Cellular Neural Networks

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***Abstract*— In this brief, the problems of delay-dependent stability analysis and strict $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\alpha$-dissipativity analysis are investigated for cellular neural networks (CNNs) with distributed delay. First, by introducing an integral partitioning technique, two new forms of Lyapunov–Krasovskii functionals are constructed, and improved distributed delay-dependent stability conditions are established in terms of linear matrix inequalities. Based on this criterion, a new sufficient delay and α-dependent condition is given to guarantee that the CNNs with distributed delay are strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\alpha$-dissipative. The results developed in this brief can tolerate larger allowable delay than existing ones in the literature, which is demonstrated by several examples.**

***Index Terms*— Cellular neural networks, dissipativity, distributed delay, integral partitioning.**

I. Introduction

Cellular neural networks (CNNs), first proposed in [1], have received considerable attention due to their extensive applications in signal processing, pattern recognition, and optimization problems [2], [3]. On the other hand, time delay is unavoidable in many biological and artificial CNNs due to the finite speed of information processing and the inherent communication time of neurons, and its existence may affect the oscillation, divergence, and stability of the system. Therefore, a great deal of attention has been devoted to the stability analysis of CNNs with time delay, see [4]–[7], for example. To mention a few, the problem of delay-dependent exponential stability analysis of delayed neural networks was investigated by using the free-weighting matrices method in [8]. By using the delay partitioning method, the conservatism of results in [8] was reduced in [9]. However, the results in [9] could be applied only to CNNs with constant delay, whereas a novel stability criterion of CNNs with interval time-varying delay by using the same technique was established in [6]. An improved results was proposed in [3] by constructing a more general Lyapunov functional based on the result in [6].

It is noted that the results mentioned above are derived for systems with discrete delays. Another type of time delay is distributed delay. Systems with distributed delay can be applied in the modeling of feeding systems and combustion chambers in a liquid monopropellant rocket motor with pressure feeding [10], [11]. Therefore, much attention has been devoted to studying neural networks with distributed delay in recent years (see some results on neural networks with infinitely distributed delay in [12] and [13]). For neural networks with finitely distributed delay, the sufficient conditions in terms of linear matrix inequalities (LMIs) are established to check the global asymptotic stability of neural networks with both multiple time-varying discrete delays and distributed delays [14]. By assuming neither differentiability nor strict monotonicity for the activation function, the analysis problem of the global exponential stability of a class of recurrent neural networks with mixed discrete and distributed delays was considered in [15]. For generalized neural networks with discrete and distributed delays, the global asymptotic stability analysis problem was solved in [16]. For CNNs, by using the integral inequality method, the problem of delay-dependent global exponential stability was studied in [17]. For stochastic neural networks with discrete and distributed time-varying delays, the exponential stability problem was investigated in [18] and [19]. Benefiting from the partitioning method, new delay-dependent stability criteria were presented for the exponential stability on stochastic neural networks with discrete interval and distributed delays in [20]. However, unlike the results for neural networks with discrete time delay, there are very few results on increasing the allowable delay for the global asymptotic stability of neural networks with distributed delay, which remains important and challenging.

Dissipative systems, introduced in [21], are very useful for a wide range of fields such as system, circuit, network, and control theory [2]. Dissipativity theory generalizes the passivity theorem, the bounded real lemma, the Kalman–Yakbovich–Popov lemma, and the circle criterion. As pointed out in [22], global dissipativity is also an important concept in dynamical neural networks. So far, the problem of dissipativity analysis for neural networks with time delay has been investigated in [22]–[24]. The dissipative property of neural networks with constant delay was analyzed in [22] and [23]. Employing Jensen's inequality and some analytical techniques, several sufficient conditions for the global dissipativity of stochastic neural networks were derived in [24]. For neural networks with infinitely distributed delay, the global dissipativity has received much attention in the literature [25]. To the best of our knowledge, few authors have considered the problem on dissipativity of neural networks with finitely distributed delay. The passivity problem of neural networks with discrete and finitely distributed time delay was addressed in [26]. The dissipativity property is more general than the passivity property, which is our second motivation.

In this brief, we aim to increase the allowable delay of existing results for stability criteria of CNNs with distributed delay systems. An improved version of distributed-delay-dependent condition in terms of LMIs is established by employing the integral partitioning technique. Based on this, a delay-dependent sufficient condition for dissipativity of CNNs which guarantees the CNNs to be stable and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\alpha$-dissipative is proposed. In addition to delay dependence, the obtained results are also dependent on the partitioning size. Finally, numerical examples are given to illustrate the effectiveness of the presented results.

Notation: $\mathbb{R}^+$ is the set of nonnegative real numbers; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $P > 0$ (≥ 0) means that P is real symmetric and positive definite

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(semidefinite); I and 0 refer to the identity matrix and zero matrix with compatible dimensions; diag$\{\cdots\}$ stands for a block diagonal matrix; A^T and A^{-1} denote the transpose and the inverse of a matrix A; $\star$ stands for the symmetric terms in a symmetric matrix and sym(A) is defined as $A + A^T$. $\mathcal{L}_2^n$ is the space of square integrable functions on $\mathbb{R}^+$ with values in $\mathbb{R}^n$; $\mathcal{L}_{2e}^n$ is the extended $\mathcal{L}_2^n$ space defined by $\mathcal{L}_{2e}^n = \{f : f$ is a measurable function on $\mathbb{R}^+$, $P_\tau f \in \mathcal{L}_2^n, \forall \tau \in \mathbb{R}^+\}$, where $(P_\tau f)(t) = f(t)$ if $t \le \tau$, and 0 if $t > \tau$. For any function $x = \{x(t)\}$, $y = \{y(t)\} \in \mathcal{L}_{2e}^n$, matrix M, we define $< x, My >_\tau = \int_0^\tau x(t)^T My(t)dt$.

II. Preliminaries

Consider the following CNNs with distributed time delay:

$$\begin{cases} \dot{x}(t) = -Ax(t) + Bf(x(t)) + A_h \int_{t-h}^{t} f(x(s))ds + u(t) \\ y(t) = f(x(t)) \\ x(t) = \varphi(t), \ \forall t \in [-2h, 0] \end{cases} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the neuron state, $u(t) \in \mathbb{R}^n$ is the input, $f(x(t)) = [f_1(x_1(t)), \ldots, f_n(x_n(t))]^T \in \mathbb{R}^n$ denotes the neuron activation function, y(t) is the output, $h > 0$ represents the system distributed delay, $\varphi(t)$ is the initial function, $A = \text{diag}\{a_1, \ldots, a_n\} > 0$, and B and A_h are known interconnection weight matrices. Throughout this brief, we assume that the activation function satisfies the following assumption.

Assumption 1: Each neuron activation function in (1), $f_i(\cdot), \ i = 1, 2, \ldots, n$ satisfies the condition

$$0 \le \frac{f_i(x) - f_i(y)}{x - y} \le k_i, \ \forall x, y \in \mathbb{R}, \ x \ne y, \ i = 1, 2, \ldots, n. \tag{2}$$

Before moving on, the following lemma and definition are required.

Lemma 1: For any matrix $M > 0$, scalars $b > a$ and $c < d \le 0$, if there exists a Lebesque vector function $\omega(s)$, then the following inequalities hold:

$$-\int_a^b \omega(s)^T M\omega(s)ds \le -\frac{1}{b-a}\tilde{\omega}^T M\tilde{\omega} \tag{3}$$

$$-\int_c^d \int_{t+\theta}^t \omega(s)^T M\omega(s)dsd\theta \le -\frac{2}{c^2 - d^2}\bar{\omega}(t)^T M\bar{\omega}(t) \tag{4}$$

where $\tilde{\omega} = \int_a^b \omega(s)ds, \ \bar{\omega}(t) = \int_c^d \int_{t+\theta}^t \omega(s)dsd\theta$.

Proof: Inequality (3) is the Jensen inequality [27]–[29], while a special version of (4) is given in [30]. For (4), the proof can be carried out following a similar line as in the proof of [27, Lemma 1]. Using Schur complement, it is easy to see that

$$\begin{bmatrix} \omega(s)^T M\omega(s) & \omega(s)^T \\ \star & M^{-1} \end{bmatrix} \ge 0$$

for any $t+c \le s \le t+d$. Integrating the above inequality over the triangle defined by $t+c \le s \le t+d$ and $c \le \theta \le d$ yields

$$\begin{bmatrix} \int_c^d \int_{t+\theta}^t \omega(s)^T M\omega(s)dsd\theta & \int_c^d \int_{t+\theta}^t \omega(s)^T dsd\theta \\ \star & \frac{c^2-d^2}{2}M^{-1} \end{bmatrix} \ge 0.$$

Using the Schur complement again, (4) holds. ■

Definition 1: Given some scalar $\alpha > 0$, matrices $\mathcal{Q}$, $\mathcal{R}$, and $\mathcal{S}$ with $\mathcal{Q}$ and $\mathcal{R}$ real symmetric, (1) is called strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\alpha$-dissipative, if for any $\tau \ge 0$, under zero initial state, the following condition is satisfied:

$$\langle y, \mathcal{Q}y\rangle_\tau + 2\langle y, \mathcal{S}u\rangle_\tau + \langle u, \mathcal{R}u\rangle_\tau \ge \alpha\langle u, u\rangle_\tau, \forall u \in \mathcal{L}_{2e}^n. \tag{5}$$

Remark 3: The left-hand side of (5) represents abstract energy supplied to system on interval $[0, \tau]$ from the external environment [31], [32]. The term α introduced on the right-hand side serves two purposes. On one hand, it makes a dissipative system strictly satisfy the dissipation inequality $\langle y, \mathcal{Q}y\rangle_\tau + 2\langle y, \mathcal{S}u\rangle_\tau + \langle u, \mathcal{R}u\rangle_\tau \ge 0$ when $u \ne 0$. On the other hand, α is also an adjustable parameter for determining the strictness of the dissipativity as defined in Definition 1. In standard definition, the system is strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$ dissipative only if there exists a scalar $\alpha > 0$ satisfying (5), however small it may be. However, in our definition, our purpose is to determine whether (5) is satisfied for a given α and whether increasing the value of α makes it harder to satisfy. Therefore, we can find the maximum allowable α such that the system is strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\alpha$-dissipative.

As in [33] and [34], we assume that $\mathcal{Q} \le 0$. Then we may write $-\mathcal{Q} = \mathcal{Q}_-^T \mathcal{Q}_-$ for some $\mathcal{Q}_-$.

Our main objective is to study the problems of stability analysis and dissipative analysis for system (1). Specifically, we are concerned with the following two problems.

1) Establish a new delay-dependent stability criteria in terms of LMIs such that system (1) is globally asymptotically stable for a given scalar h.
2) Establish a sufficient condition in terms of LMIs such that system (1) is asymptotically stable and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\alpha$-dissipative for a given scalar h.

III. Main Results

In this brief, we first investigate the integral partitioning method to partition the integral interval of the neuron activation function into m equal parts, i.e., $[t-h, t-(\frac{m-1}{m})h]$, $[t-(\frac{m-1}{m})h, t-(\frac{m-2}{m})h], \ldots, [t-(\frac{h}{m}), t]$, for systems with distributed delay in order to further increase the allowable delay of the existing stability results.

In this section, an improved sufficient condition is derived first by employing the integral partitioning technique, which guarantees that the system in (1) is globally asymptotically stable.

Theorem 3: For a given scalar h and integer $m > 0$, the system in (1) is globally asymptotically stable, if there exist matrices $P > 0$, $Q > 0$, $Z > 0$, $R > 0$, $\left[\begin{smallmatrix} M & S \\ \star & N \end{smallmatrix}\right] > 0$, and $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \ge 0$ and $L = \text{diag}\{l_1, l_2, \ldots, l_n\} \ge 0$ such that the following LMI holds:

$$\Omega < 0 \tag{6}$$

where

$$\begin{aligned} \Omega = \text{sym}(&W_P^T PW_S + W_\Lambda^T \Lambda W_S + W_P^T KLW_\Lambda \\ &- W_\Lambda^T LW_\Lambda) + W_Q^T \bar{Q}W_Q + W_Z^T \bar{Z}W_Z \\ &+ W_R^T \bar{R}W_R + \frac{h}{m}W_M^T \bar{M}W_M - \frac{m}{h}W_N^T \bar{M}W_N \end{aligned}$$

$$W_P = \begin{bmatrix} I_n & 0_{n,(2m+3)n} \end{bmatrix}$$
$$W_\Lambda = \begin{bmatrix} 0_{n,(m+2)n} & I_n & 0_{n,(m+1)n} \end{bmatrix}$$
$$W_S = \begin{bmatrix} -A & 0_{n,(m+1)n} & B & \overbrace{A_h \ \ldots \ A_h}^{m} & 0_{n,n} \end{bmatrix}$$
$$W_Q = \begin{bmatrix} 0_{mn,n} & I_{mn} & 0_{mn,(m+3)n} \\ \hline 0_{mn,2n} & I_{mn} & 0_{mn,(m+2)n} \end{bmatrix}$$
$$W_Z = \begin{bmatrix} 0_{mn,(m+3)n} & I_{mn} & 0_{mn,n} \\ \hline 0_{mn,(m+4)n} & I_{mn} & \end{bmatrix}$$
$$W_R = \begin{bmatrix} -A & 0_{n,(m+1)n} & B & \overbrace{A_h \ \ldots \ A_h}^{m} & 0_{n,n} \\ \hline \frac{h}{m} I_n & -I_n & 0_{n,(2m+2)n} & & \end{bmatrix}$$
$$W_M = \begin{bmatrix} I_n & 0_{n,(2m+3)n} & \\ \hline 0_{n,(m+2)n} & I_n & 0_{n,(m+1)n} \end{bmatrix}$$
$$W_N = \begin{bmatrix} 0_{n,n} & I_n & 0_{n,(2m+2)n} \\ \hline 0_{n,(m+3)n} & I_n & 0_{n,mn} \end{bmatrix}$$
$$\bar{Q} = \begin{bmatrix} Q & 0_{mn,mn} \\ \star & -Q \end{bmatrix}, \ \bar{Z} = \begin{bmatrix} \frac{h}{m} Z & 0_{n,n} \\ \star & -\frac{m}{h} Z \end{bmatrix}$$
$$\bar{R} = \begin{bmatrix} \frac{1}{2}(\frac{h}{m})^2 R & 0_{n,n} \\ \star & -2(\frac{m}{h})^2 R \end{bmatrix}, \ \bar{M} = \begin{bmatrix} M & S \\ \star & N \end{bmatrix}$$
$$K = \mathrm{diag}\{k_1, k_2, \ldots, k_n\}.$$

Proof: Construct a new Lyapunov functional candidate as

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) + V_4(x(t)) \quad (7)$$

where

$$V_1(x(t)) = x(t)^T P x(t) + 2\sum_{i=1}^{n} \lambda_i \int_0^{x_i(t)} f_i(s) ds$$
$$V_2(x(t)) = \int_{t-\frac{h}{m}}^{t} [\eta(\theta)^T Q \eta(\theta) + \varphi(\theta)^T Z \varphi(\theta)] d\theta$$
$$V_3(x(t)) = \int_{-\frac{h}{m}}^{0} \int_{t+\theta}^{t} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \begin{bmatrix} M & S \\ \star & N \end{bmatrix} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds d\theta$$
$$V_4(x(t)) = \int_{-\frac{h}{m}}^{0} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{x}(s)^T R \dot{x}(s) ds d\lambda d\theta$$

with

$$\eta(t) = \begin{bmatrix} \int_{t-\frac{h}{m}}^{t} x(s)ds \\ \vdots \\ \int_{t-\frac{(m-1)h}{m}}^{t-\frac{(m-2)h}{m}} x(s)ds \\ \int_{t-h}^{t-\frac{(m-1)h}{m}} x(s)ds \end{bmatrix}, \ \varphi(t) = \begin{bmatrix} \int_{t-\frac{h}{m}}^{t} f(x(s))ds \\ \vdots \\ \int_{t-\frac{(m-1)h}{m}}^{t-\frac{(m-2)h}{m}} f(x(s))ds \\ \int_{t-h}^{t-\frac{(m-1)h}{m}} f(x(s))ds. \end{bmatrix}$$

Evaluating the derivative of $V(x(t))$ in (7) along the solutions of (1), we obtain

$$\dot{V}(x(t)) = \dot{V}_1(x(t)) + \dot{V}_2(x(t)) + \dot{V}_3(x(t)) + \dot{V}_4(x(t)) \quad (8)$$

where

$$\dot{V}_1(x(t)) = 2x(t)^T P \dot{x}(t) + 2f(x(s))^T \Lambda \dot{x}(t) \quad (9)$$
$$\dot{V}_2(x(t)) = \eta(t)^T Q \eta(t) - \eta\left(t - \frac{h}{m}\right)^T Q \eta\left(t - \frac{h}{m}\right) + \varphi(t)^T Z \varphi(t) - \varphi\left(t - \frac{h}{m}\right)^T Z \varphi\left(t - \frac{h}{m}\right) \quad (10)$$
$$\dot{V}_3(x(t)) = \frac{h}{m} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \bar{M} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} - \int_{t-\frac{h}{m}}^{t} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \bar{M} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds \quad (11)$$
$$\dot{V}_4(x(t)) = \frac{1}{2}\left(\frac{h}{m}\right)^2 \dot{x}(t)^T R \dot{x}(t) - \int_{-\frac{h}{m}}^{0} \int_{t+\theta}^{t} \dot{x}(s)^T R \dot{x}(s) ds d\theta. \quad (12)$$

By Lemma 1, we have

$$\int_{t-\frac{h}{m}}^{t} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \bar{M} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds \le -\frac{m}{h} \bar{x}_1(t)^T \bar{M} \bar{x}_1(t) \quad (13)$$

$$-\int_{-\frac{h}{m}}^{0} \int_{t+\theta}^{t} \dot{x}(s)^T R \dot{x}(s) ds d\theta \le -2\left(\frac{m}{h}\right)^2 \bar{x}_2(t)^T R \bar{x}_2(t) \quad (14)$$

where

$$\bar{x}_1(t) = \int_{t-\frac{h}{m}}^{t} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds, \ \bar{x}_2(t) = \int_{-\frac{h}{m}}^{0} \int_{t+\theta}^{t} \dot{x}(s) ds d\theta.$$

On the other hand, from (2) we can get

$$f_i(x_i(t))[f_i(x_i(t)) - k_i x_i(t)] \le 0, \qquad i = 1, 2, \ldots, n$$

which derives

$$2f(x(t))^T L K x(t) - 2f(x(t))^T L f(x(t)) \ge 0 \quad (15)$$

for any $L = \mathrm{diag}\{l_1, l_2, \ldots, l_n\} \ge 0$.

Substituting (9)–(15) into (8) yields $\dot{V}(x(t)) \le \zeta(t)^T \Omega \zeta(t)$ where

$$\zeta(t) = \begin{bmatrix} x(t)^T & \eta(t)^T & \int_{t-\frac{(m+1)h}{m}}^{t-h} x(s)^T ds & f(x(t))^T & \varphi(t)^T & \int_{t-\frac{(m+1)h}{m}}^{t-h} f(x(s))^T ds \end{bmatrix}^T \in \mathbb{R}^{2(m+2)n}.$$

Therefore, if $\Omega < 0$, $\dot{V}(x(t)) < 0$ is derived and (1) is globally asymptotically stable. This completes the proof. ∎

The main technique utilized in this brief is the integral partitioning idea, which partitions the integral interval into m equal subintervals.

Remark 4: For the maximum allowable distributed delay, it is computed with bisection method by running the program with different values of h.

In order to further increase the allowable distributed delay, we also give the following more general theorem.

Theorem 4: For a given scalar h and integer $m > 0$, the system in (1) is globally asymptotically stable, if there exist matrices $P > 0, \begin{bmatrix} Q & V \\ \star & Z \end{bmatrix} > 0$, $R_j > 0$, $\begin{bmatrix} M_j & S_j \\ \star & N_j \end{bmatrix} > 0$,

$j = 1, 2, \ldots, m$ and $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \geq 0$ and $L = \text{diag}\{l_1, l_2, \ldots, l_n\} \geq 0$ such that the following LMI holds:

$$\tilde{\Omega} < 0 \tag{16}$$

where

$$\begin{aligned}
\tilde{\Omega} = \text{sym}\Big(&W_P^T P W_S + W_\Lambda^T \Lambda W_S + W_P^T K L W_\Lambda \\
&- W_\Lambda^T G W_\Lambda\Big) + W_{QZ}^T \tilde{Q} W_{QZ} \\
&+ \sum_{j=1}^m \frac{h^2(2j-1)}{2m^2} W_S^T R_j W_S \\
&- \sum_{j=1}^m \frac{2m^2}{h^2(2j-1)} W_{Rj}^T R_j W_{Rj} \\
&+ \sum_{j=1}^m \frac{h}{m} W_M^T \tilde{M}_j W_M - \sum_{j=1}^m \frac{m}{h} W_{Mj}^T \tilde{M}_j W_{Mj}
\end{aligned}$$

$$\begin{aligned}
W_P &= \begin{bmatrix} I_n & 0_{n,(2m+3)n} \end{bmatrix} \\
W_\Lambda &= \begin{bmatrix} 0_{n,(m+2)n} & I_n & 0_{n,(m+1)n} \end{bmatrix} \\
W_S &= \begin{bmatrix} -A & 0_{n,(m+1)n} & B & \overbrace{A_h \ \ldots \ A_h}^{m} & 0_{n,n} \end{bmatrix} \\
W_{QZ} &= \begin{bmatrix} 0_{mn,n} \quad I_{mn} \quad 0_{mn,(m+3)n} \\ \hline 0_{mn,(m+3)n} \quad I_{mn} \quad 0_{mn,n} \\ \hline 0_{mn,2n} \quad I_{mn} \quad 0_{mn,(m+2)n} \\ \hline 0_{mn,(m+4)n} \quad I_{mn} \end{bmatrix} \\
W_M &= \begin{bmatrix} I_n \quad 0_{n,(2m+3)n} \\ \hline 0_{n,(m+2)n} \quad I_n \quad 0_{n,(m+1)n} \end{bmatrix} \\
\bar{Q} &= \begin{bmatrix} Q & V \\ \star & Z \end{bmatrix}, \ \tilde{Q} = \begin{bmatrix} \bar{Q} & 0_{2mn,2mn} \\ \star & -\bar{Q} \end{bmatrix} \\
W_{Rj} &= \begin{bmatrix} \frac{h}{m} I_n & 0_{n,(j-1)n} & -I_n & 0_{n,(2m+3-j)n} \end{bmatrix} \\
W_{Mj} &= \begin{bmatrix} 0_{n,jn} \quad I_n \quad 0_{n,(2m+3-j)n} \\ \hline 0_{n,(m+2+j)n} \quad I_n \quad 0_{n,(m+1-j)n} \end{bmatrix} \\
\tilde{M}_j &= \begin{bmatrix} M_j & S_j \\ \star & N_j \end{bmatrix}, \ j = 1, 2, \ldots, m \\
K &= \text{diag}\{k_1, k_2, \ldots, k_n\}.
\end{aligned}$$

Proof: We introduce a new Lyapunov functional as follows:

$$\tilde{V}(x(t)) = V_1(x(t)) + \tilde{V}_2(x(t)) + \tilde{V}_3(x(t)) + \tilde{V}_4(x(t)) \tag{17}$$

where $V_1(x(t))$ is defined in (7) and

$$\begin{aligned}
\tilde{V}_2(x(t)) &= \int_{t-\frac{h}{m}}^t \begin{bmatrix} \eta(\theta) \\ \varphi(\theta) \end{bmatrix}^T \begin{bmatrix} Q & V \\ \star & Z \end{bmatrix} \begin{bmatrix} \eta(\theta) \\ \varphi(\theta) \end{bmatrix} d\theta \\
\tilde{V}_3(x(t)) &= \sum_{j=1}^m \int_{-\frac{jh}{m}}^{-\frac{(j-1)h}{m}} \int_{t+\theta}^t \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \begin{bmatrix} M_j & S_j \\ \star & N_j \end{bmatrix} \\
&\quad \times \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds d\theta \\
\tilde{V}_4(x(t)) &= \sum_{j=1}^m \int_{-\frac{jh}{m}}^{-\frac{(j-1)h}{m}} \int_\theta^0 \int_{t+\lambda}^t \dot{x}^T(s) R_j \dot{x}(s) ds d\lambda d\theta.
\end{aligned}$$

The derivatives of $\tilde{V}_i(t), i = 2, 3, 4$, are given by

$$\begin{aligned}
\dot{\tilde{V}}_2(x(t)) &= \begin{bmatrix} \eta(t) \\ \varphi(t) \end{bmatrix}^T \begin{bmatrix} Q & V \\ \star & Z \end{bmatrix} \begin{bmatrix} \eta(t) \\ \varphi(t) \end{bmatrix} \\
&\quad - \begin{bmatrix} \eta(t-\frac{h}{m}) \\ \varphi(t-\frac{h}{m}) \end{bmatrix}^T \begin{bmatrix} Q & V \\ \star & Z \end{bmatrix} \begin{bmatrix} \eta(t-\frac{h}{m}) \\ \varphi(t-\frac{h}{m}) \end{bmatrix}
\end{aligned} \tag{18}$$

$$\begin{aligned}
\dot{\tilde{V}}_3(x(t)) &= \sum_{j=1}^m \frac{h}{m} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \tilde{M}_j \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \\
&\quad - \sum_{j=1}^m \int_{t-\frac{jh}{m}}^{t-\frac{(j-1)h}{m}} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \tilde{M}_j \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds
\end{aligned} \tag{19}$$

$$\begin{aligned}
\dot{\tilde{V}}_4(x(t)) &= \sum_{j=1}^m \frac{h^2[j^2-(j-1)^2]}{2m^2} \dot{x}(t)^T R_j \dot{x}(t) \\
&\quad - \sum_{j=1}^m \int_{-\frac{jh}{m}}^{-\frac{(j-1)h}{m}} \int_{t+\theta}^t \dot{x}(s)^T R_j \dot{x}(s) ds d\theta.
\end{aligned} \tag{20}$$

Similarly, by using Lemma 1, we have

$$-\int_{t-\frac{jh}{m}}^{t-\frac{(j-1)h}{m}} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix}^T \tilde{M}_j \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds \leq \tilde{x}_j(t)^T \tilde{M}_j \tilde{x}_j(t) \tag{21}$$

$$\begin{aligned}
&-\int_{-\frac{jh}{m}}^{-\frac{(j-1)h}{m}} \int_{t+\theta}^t \dot{x}(s)^T R_j \dot{x}(s) ds d\theta \\
&\quad \leq -\frac{2m^2}{h^2(2j-1)} r_j(t)^T R_j r_j(t)
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
\tilde{x}_j(t) &= \int_{t-\frac{jh}{m}}^{t-\frac{(j-1)h}{m}} \begin{bmatrix} x(s) \\ f(x(s)) \end{bmatrix} ds \\
r_j(t) &= \frac{h}{m} x(t) - \int_{t-\frac{jh}{m}}^{t-\frac{(j-1)h}{m}} x(s) ds.
\end{aligned}$$

Substituting (21)–(22) into (19)–(20) and combining (9), (15), and (18) yields

$$\dot{\tilde{V}}(x(t)) \leq \zeta(t)^T \tilde{\Omega} \zeta(t).$$

Therefore, if (6) holds, then $\dot{\tilde{V}}(x(t)) < 0$, which guarantees that (1) is globally asymptotically stable. ■

Remark 5: The proposed Lyapunov functional in (17) is more general than that in (7) for two reasons. On one hand, we generalize the matrix $\left[\begin{smallmatrix} Q & 0 \\ \star & Z \end{smallmatrix}\right]$ in (7) by matrix $\left[\begin{smallmatrix} Q & V \\ \star & Z \end{smallmatrix}\right]$ in (17). On the other hand, motivated by the idea in [3], we confine the matrices $\tilde{M}_j$ and $\tilde{R}_j$ on multiple subintervals in (17) not just one subinterval $[-h/m, 0]$ in (7).

Next, we will extend our results to the problem of dissipativity analysis for CNNs with distributed time delay.

Theorem 5: Let scalar $\alpha > 0$ and the matrices $\mathcal{Q}$, $\mathcal{S}$, and $\mathcal{R}$ be given with $\mathcal{Q}$ and $\mathcal{R}$ real symmetric. Then, for a given scalar h and integer $m > 0$, the system in (1) is globally asymptotically stable and strictly $(\mathcal{Q}, \mathcal{S}, \mathcal{R})$-$\alpha$-dissipative, if there exist matrices $P > 0$, $\left[\begin{smallmatrix} Q & V \\ \star & Z \end{smallmatrix}\right] > 0$, $R_j > 0$, $\left[\begin{smallmatrix} M_j & S_j \\ \star & N_j \end{smallmatrix}\right] > 0$,

TABLE I
MAXIMUM ALLOWABLE DISTRIBUTED DELAY h OBTAINED BY DIFFERENT METHODS

Methods	[17]	[20]	$m=1$		$m=2$		$m=4$	
			Theorem 1	Theorem 2	Theorem 1	Theorem 2	Theorem 1	Theorem 2
h_{max}	1.2480	1.2480	2.1828	2.1828	2.6754	2.7384	3.0422	3.4949

$j=1,2,\ldots,m$ and $\Lambda=\text{diag}\{\lambda_1,\lambda_2,\ldots,\lambda_n\}\geq 0$ and $L=\text{diag}\{l_1,l_2,\ldots,l_n\}\geq 0$ such that the following LMI holds:

$$\breve{\Omega}<0 \tag{23}$$

where

$$\begin{aligned}
\breve{\Omega} &= \text{sym}(\breve{W}_P^T P\breve{W}_S+\breve{W}_\Lambda^T\Lambda\breve{W}_S+\breve{W}_P^T KL\breve{W}_\Lambda \\
&\quad -\breve{W}_\Lambda^T G\breve{W}_\Lambda)+\breve{W}_{QZ}^T\tilde{Q}\breve{W}_{QZ} \\
&\quad +\sum_{j=1}^{m}\frac{h^2(2j-1)}{2m^2}\breve{W}_S^T R_j\breve{W}_S \\
&\quad -\sum_{j=1}^{m}\frac{2m^2}{h^2(2j-1)}\breve{W}_{Rj}^T R_j\breve{W}_{Rj} \\
&\quad +\sum_{j=1}^{m}\frac{h}{m}\breve{W}_M^T\tilde{M}_j\breve{W}_M-\sum_{j=1}^{m}\frac{m}{h}\breve{W}_{Mj}^T\tilde{M}_j\breve{W}_{Mj} \\
&\quad -\breve{W}_\Lambda^T\mathcal{Q}\breve{W}_\Lambda-sym(\breve{W}_U^T\mathcal{S}\breve{W}_\Lambda) \\
&\quad -\breve{W}_U^T(\mathcal{R}-\alpha I)\breve{W}_U \\
\breve{W}_P &= \begin{bmatrix}W_P & 0_{n,n}\end{bmatrix},\ \breve{W}_\Lambda=\begin{bmatrix}W_\Lambda & 0_{n,n}\end{bmatrix} \\
\breve{W}_S &= \begin{bmatrix}W_S & I_n\end{bmatrix},\ \breve{W}_{QZ}=\begin{bmatrix}W_{QZ} & 0_{4mn,n}\end{bmatrix} \\
\breve{W}_M &= \begin{bmatrix}W_M & 0_{2n,n}\end{bmatrix},\ \breve{W}_U=\begin{bmatrix}0_{n,(2m+4)n} & I_n\end{bmatrix} \\
\breve{W}_{Rj} &= \begin{bmatrix}W_{Rj} & 0_{n,n}\end{bmatrix},\ \breve{W}_{Mj}=\begin{bmatrix}W_{Mj} & 0_{2n,n}\end{bmatrix} \\
K &= \text{diag}\{k_1,k_2,\ldots,k_n\}.
\end{aligned}$$

Proof: The inequality in (6) can be derived by (23), therefore, the system in (1) is stable. To establish the dissipativity performance, we assume zero initial state condition, and have $\tilde{V}(x(0))=0$. Then we introduce the following cost function for $\tau>0$:

$$J(\tau,\alpha)=\int_0^\tau\Big[y(t)^T\mathcal{Q}y(t)+2y(t)^T\mathcal{S}u(t)+u(t)^T(\mathcal{R}-\alpha I)u(t)\Big]dt.$$

Now, we have

$$\begin{aligned}
&\dot{\tilde{V}}(x(t))-y(t)^T\mathcal{Q}y(t)-2y(t)^T\mathcal{S}u(t)-u(t)^T(\mathcal{R}-\alpha I)u(t) \\
\leq\ &\breve{\zeta}(t)^T\breve{\Omega}\breve{\zeta}(t)<0
\end{aligned} \tag{24}$$

where

$$\breve{\zeta}(t)=\begin{bmatrix}\zeta(t)\\ u(t)\end{bmatrix}.$$

Integrating (24) from 0 to τ gives

$$\int_0^\tau\dot{\tilde{V}}(x(t))dt=\tilde{V}(x(\tau))-\tilde{V}(x(0))\leq J(\tau,\alpha) \tag{25}$$

which implies that the condition in (5) holds. Therefore, the system in (1) is dissipative and the proof is completed. ■

Remark 6: The conditions obtained in Theorem 5 depend on not only the distributed delay h but also scalar α which can represent the dissipative margin.

TABLE II
MAXIMUM ALLOWABLE DISTRIBUTED DELAY h AND α

Methods	Theorem 5			
	$m=1$	$m=2$	$m=3$	$m=4$
h_{max} with $\alpha=1$	3.6793	4.1242	4.5722	4.9760
α_{max} with $h=3.5$	1.8496	1.9708	1.9877	1.9931

TABLE III
MAXIMUM ALLOWABLE DISTRIBUTED DELAY h OBTAINED BY DIFFERENT METHODS

Methods	[26], [36]	Theorem 5			
		$m=1$	$m=2$	$m=3$	$m=4$
h_{max}	3.8571	4.0242	4.3759	4.7652	5.1405

Remark 7: When $\mathcal{Q}=0$, $\mathcal{S}=I$, and $\mathcal{R}=2\alpha I$ in Theorem 5, we can obtain the corresponding strictly passive results which satisfy $2\langle y,\mathcal{S}u\rangle_\tau\geq-\alpha\langle u,u\rangle_\tau$. Some passivity results for CNNs can be found in [26] and [35].

IV. ILLUSTRATIVE EXAMPLES

In this section, some examples are provided to illustrate the applicability and efficiency of the proposed approach.

Example 3: Consider the following distributed delay CNN in (1) with $u(t)=0$:

$$A=\begin{bmatrix}2 & 0\\ 0 & 0.9\end{bmatrix},\ B=\begin{bmatrix}0.1 & 0.4\\ 0 & 0.5\end{bmatrix}$$
$$A_h=\begin{bmatrix}-2 & 0.5\\ -2 & -2\end{bmatrix},\ K=\begin{bmatrix}0.5 & 0\\ 0 & 0.3\end{bmatrix}.$$

The maximum allowable distributed delay h satisfying (6) and (16) can be calculated by using some standard LMI solver. Table I presents a comparison which shows that larger allowable delays can be obtained using our approach.

Example 4: Consider a distributed delay CNN in (1) with the following parameters:

$$A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix},\ A_h=\begin{bmatrix}-1 & 0\\ 0.3 & -1\end{bmatrix},\ B=\begin{bmatrix}-1 & 0.5\\ 0 & -1\end{bmatrix}$$
$$\mathcal{Q}=-I,\ \mathcal{S}=\begin{bmatrix}1 & 0\\ 0.3 & 1\end{bmatrix},\ \mathcal{R}=3I,\ K=\begin{bmatrix}0.5 & 0\\ 0 & 0.7\end{bmatrix}.$$

The maximum allowable distributed delay h satisfying (23) can be calculated by using some standard LMI solver. Table II lists the maximum allowable h for a given α and the maximum α for a given h by using the method in Theorem 5. It is seen from Table II that much larger values of h and α can be obtained by using Theorem 5 in this brief.

In order to demonstrate the improvement of our results, we compare our passivity results with those in [26] and [36]. Based on Remark 7, we choose

$$\mathcal{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ \mathcal{S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \alpha = 0.5,\ \mathcal{R} = 2\alpha I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Table III gives the comparison results for maximum allowable distributed delay h with the above given parameters.

V. Conclusion

In this brief, the problems of stability and dissipativity analysis for CNNs with finite distributed delay were investigated. The delay-dependent stability conditions in terms of LMIs were proposed by employing the integral partitioning method. Based on this, we extended the method to solve the dissipativity analysis problem. Finally, some examples were given to demonstrate the effectiveness and applicability of our methods.

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